Frequency Modulation Explained

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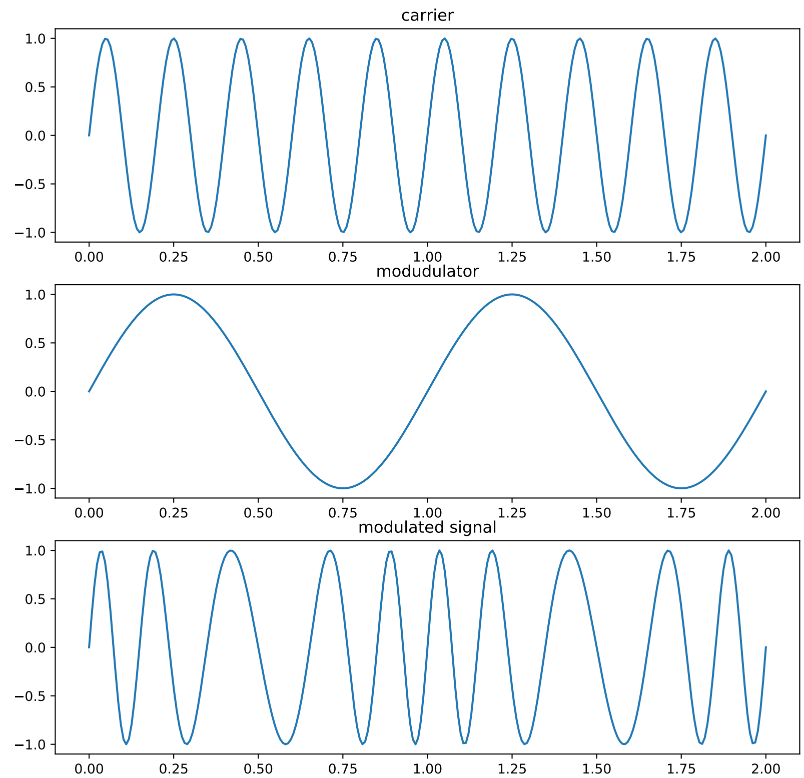
1 Introduction

The frequency modulation synthesis has been made available quite soon by modular synthesizers where the VCO (voltage control oscillator) can be controlled by an other oscillator. In 1967, John Chowning developed further the concept and allowed the control of the temporal evolution of the spectral components of a musical sound. This kind of synthesis has been popularized by Yamaha in 1983 with the DX7, a digital synthesizer which uses this type of synthesis. It is now a common synthesis available on some synthesizers, including the Yamaha Montage. We will see that this synthesis can generate multiple harmonics with few operators and is quite flexible (one output level per operator : 6 on a DX7, 8 on a Montage).

The main idea is to take two signals : a carrier and a modulator. The carrier signal is outputted with a time shift proportional with the modulation signal[1].

The following example shows a carrier, a modulator and the modulated signal. We can see that when the slope of the modulator is upward, the increasing phase shift makes the frequency higher, and when the slope is downward, the decreasing phase shift makes the frequency lower.

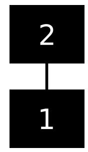
[1] The described modulation is more accurately a phase modulation, but frequency modulation is more commonly used by language abuse.



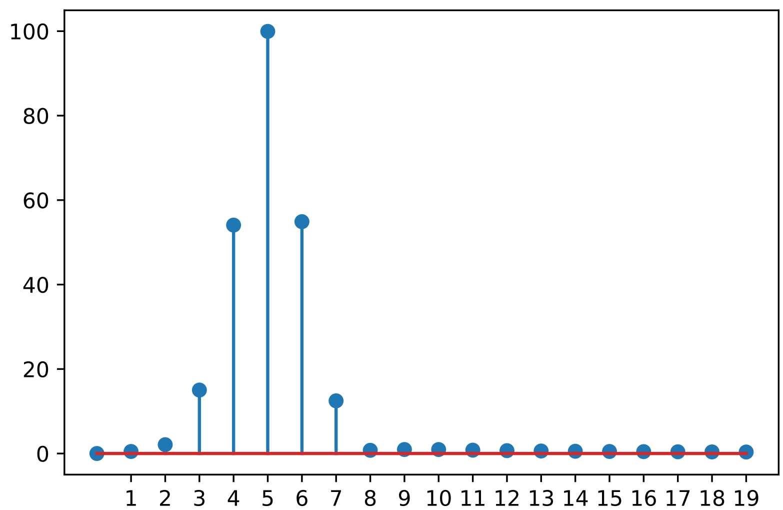
However, when analysing a sound produced by frequency modulation, the best way is to analyze the produced harmonics since the ear decompose it this way.

2 Harmonics produced by FM synthesis : 2 operators

Lets take a simple case : a sine function modulated by an other sine function. Lets play a note with the frequency f, use a carrier frequency $f_c = 5 \times f$ and a modulator frequency $f_m = 1 \times f$. On a DX7 or a Yamaha Montage, this will typically use an algorithm represented by a stack with the carrier at the bottom (1), which is modulated by the operator just above (2) :

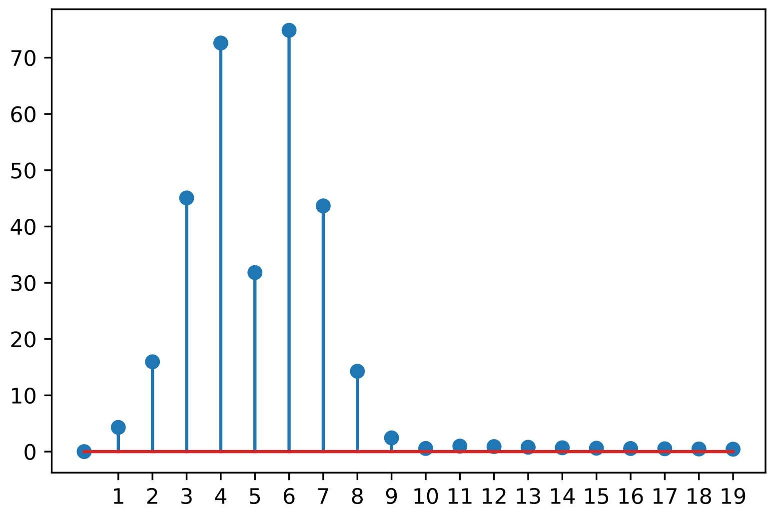


The produced harmonics are represented in the following graph (on the X axis, we have the harmonic ranks which are equal to the harmonic frequency divided by the note frequency).

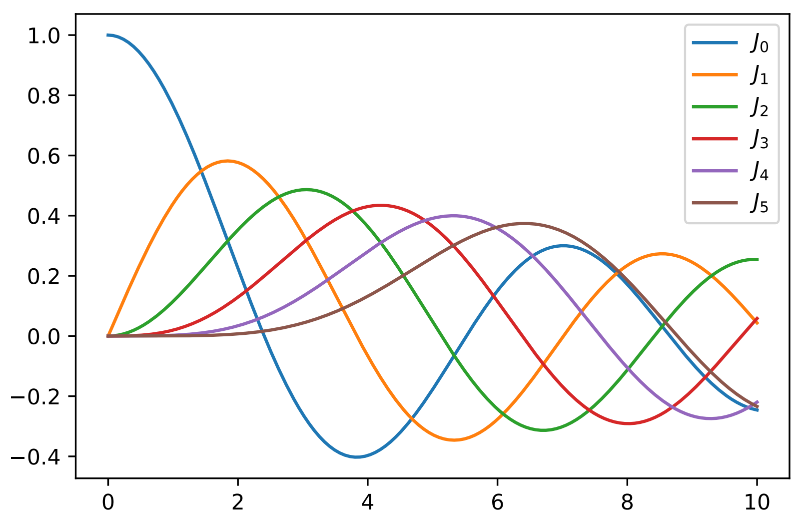


We can see a signal where a main harmonic is at the carrier frequency and other harmonics are put at distances which are multiple of the modulator frequency. Then, the frequencies are $f_n = f_c + n \times f_m$ where n can be negative.

Let's increase the output level of the modulator, we now have a wider spectrum :



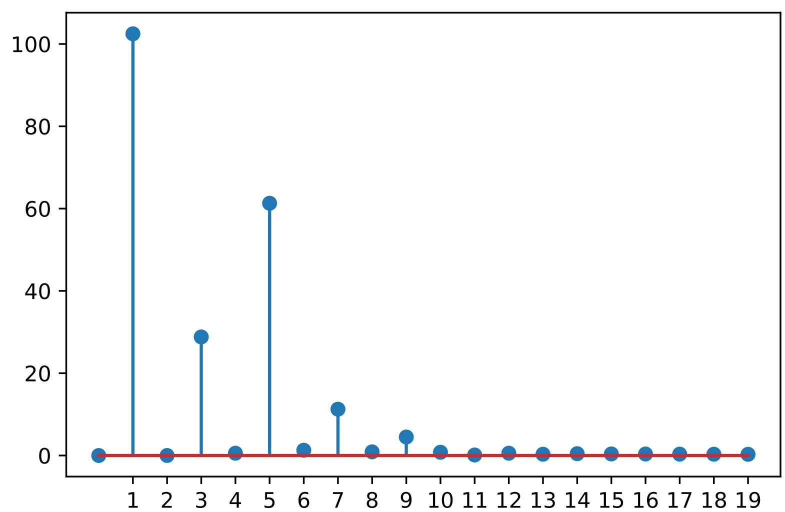
We can also see that the center harmonic is reduced. In fact, the amplitudes of the different harmonics are given by the Bessel functions (J_n) which are represented by the curves :



On the X axis, we have the modulation index which is related to the output level of the modulator. The different curves give the different amplitudes of the harmonics. Then a harmonic at the frequency $f_n = f_c + n \times f_m$ will have an amplitude equal to $J_n(\beta)$ where β is the modulation index. Note that β is related to the modulator output level : the higher the level, the higher the modulator index is, but they are not equal. We usually have and exponential curve[2] which makes it possible to reach high modulation indexes (13.12 on a DX7) and a good resolution with low indexes (used to correct the timbre of the sound).

We can see that all $J_n(0)$ are null excepted $J_0(0) = 1$. This is logical : if the modulator is null, the carrier harmonic will be unchanged. When β increases we have more and more harmonics, but all Bessel curves oscillates then we may have a spectrum with weak harmonics in a way difficult to anticipate. It is the joy of frequency modulation! In an approximate way, for a given modulation index β, $J_n(\beta)$ begins to be weak for n greater than β. The produced harmonics are then between $f_c - \beta \times f_m$ and $f_c + \beta \times f_m$.

Now, let's try a modulation of a carrier with a frequency $f_c = 1 \times f$ by a modulator at $f_m = 2 \times f$.



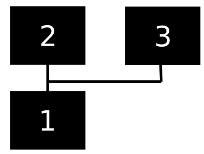
Since all harmonics are at a frequency $f_n = f_c + n \times f_m = (1+2n)f$, all harmonics are odd. This property is interesting since the clarinet and other reed instruments (with a cylindrical bore) have weak even harmonics. This doesn't mean a clarinet can be imitated accurately with only 2 operators, but using an odd carrier and even modulators is surely required to do so.

With such a configuration, if n is negative, the harmonic is not lost : we take the absolute value of the frequency to get the actual frequency. If two harmonics have the same frequency, the amplitudes are added, but some phase inversion may require the addition of a negative and a positive value.

[2] On a DX7, the relationship is $\beta = \pi \times 2^{\frac{33}{16} - \frac{99-\text{level}}{8}}$.

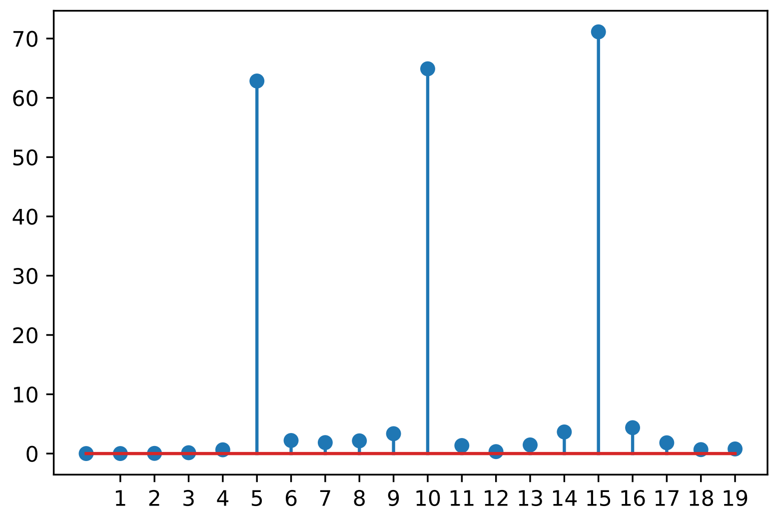
3 Let's add a modulator

On a DX7 and on a Montage, we can stack operators, but also add the signal from two operators. Added operators are put on the same level. Then, the following algorithm have 2 operators, 2 and 3 which are added. The resulting signal modulates a third operator, 1.

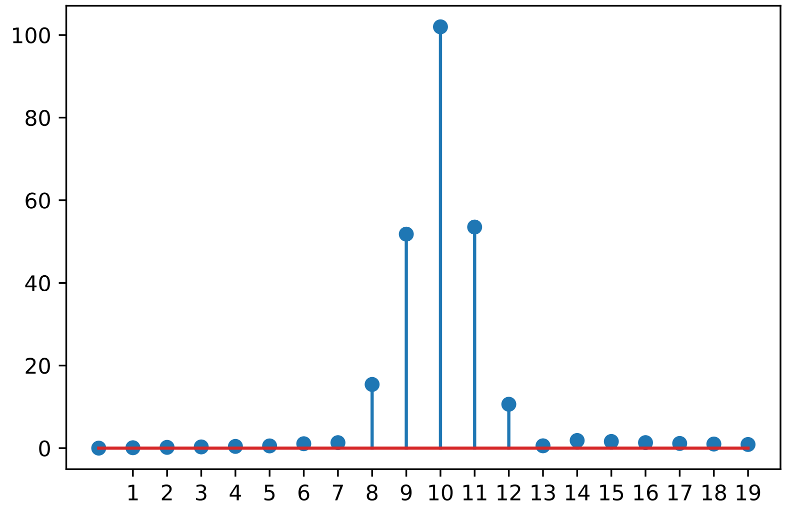


Let's take a carrier (operator 1) with a frequency $10f$, a modulator (operator 2) with a frequency $5f$ and a second modulator (operator 3) with a frequency $1f$. The index modulation of the operator 3 is low in order to avoid too many harmonics.

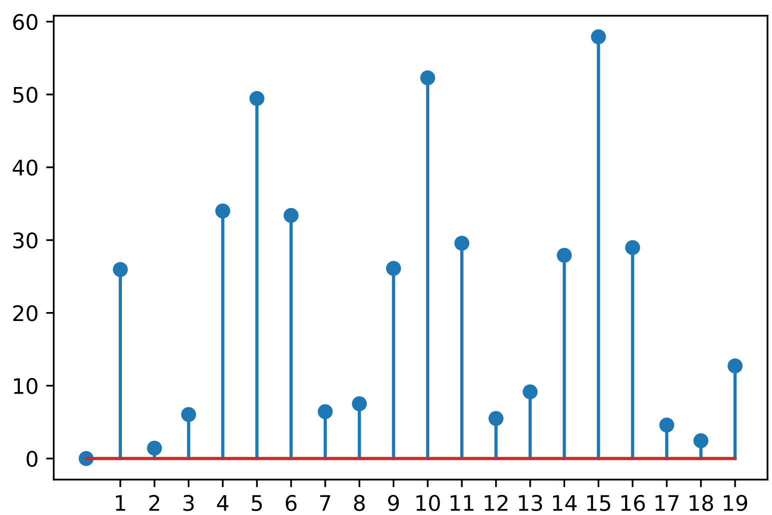
With only the operators 1 and 2, we have the spectrum :



With only the operators 1 and 3, we have the spectrum :



And with all three operators, we find the following spectrum :

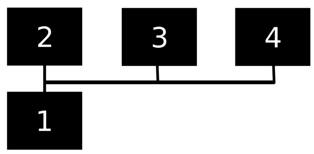


We can see that each of the harmonics of the first spectrum is replaced by the pattern of the second spectrum.

Then the carrier (operator 1) at $10f$ is modulated by the operator 2 ($5f$) which gives the harmonics $(10-5)f$, $10f$, $(10+5)f$. Each of these harmonics is modulated by the operator 3. For example, the harmonic at frequency $(10-5)f$ gives harmonics $(10-5-1)f$, $(10-5)f$, $(10-5+1)f$.

Let's sum up. The carrier is modulated twice. With a low modulation index, each of these modulations multiplies by 3 the number of harmonics and we have roughly $3 \times 3 = 9$ harmonics. With higher modulation indexes, we can have quickly an important number of harmonics. However, in most cases, the frequency factors are low then many harmonic frequencies coincide and their amplitudes are added.

We can go further, with 3 modulators and 1 carrier :



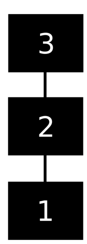
Here, even if all modulators create only 3 harmonics (with a low modulation index), the number of harmonics is multiplied at each modulation which gives $3 \times 3 \times 3 = 27$ harmonics.

Why should we generate 27 harmonics when most of them will overlap ? One answer is that with the modulators $1 \times f$, $2 \times f$ and $3 \times f$, and a small modulation indexe β (≈ 1), we can generate the harmonics $f_c + 1 \times (-1/0/1)f + 2 \times (-1/0/1)f + 3 \times (-1/0/1)f$. If we develop, $f_c + 0 \times f$ will be present 3 times, $f_c + 1 \times f$ will be present 3 times, $f_c + 2 \times f$ will be present 3 times, $f_c + 3 \times f$ will be present 2 times, $f_c + 4 \times f$ will be present 2 times, $f_c + 5 \times f$ will be present 1 time, $f_c + 6 \times f$ will be present 1 time and $f_c + 7 \times f$ will be present 1 time. Then the spectrum will decrease quite smoothly in comparison with the $J_n(\beta)$ decrease for a given modulation index β.

This is quite complex, which could explain why the DX7 propose only two algorithms with 3 modulators on a single carrier.

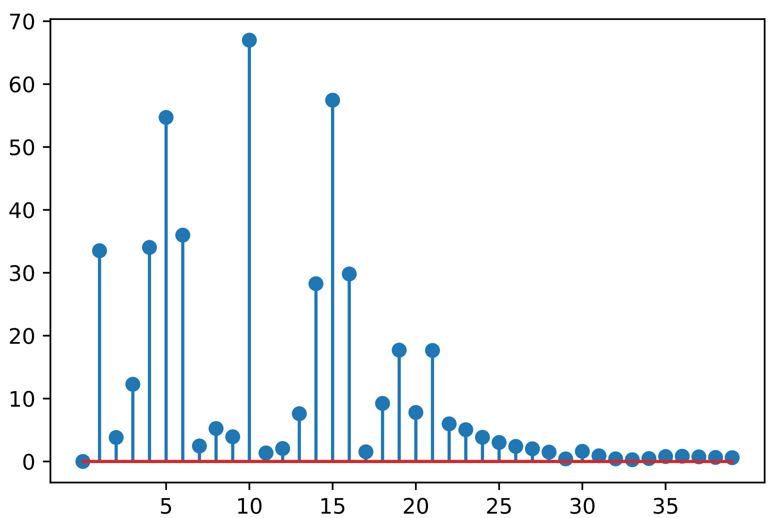
4 Let's stack modulators

Now, let's use an operator 3 to modulate the operator 2. The output of this one will modulate operator 1 (carrier) :



We can apply what has been seen to operators 3 and 2. The result is a set of harmonics. Then we have to imagine that each of these harmonics acts as a single operator like in the previous section... then we can expect an huge combinational of harmonics !

We can view the situation from an other point of view : the carrier is modulated by the operator 2 (as it would be with 2 operators), then each harmonic is modulated again by the operator 3, but with a modulation index proportional to the harmonic rank[3]. The higher the rank n, the more the harmonic is modulated and will generate a spreaded part of spectrum. This can be illustrated by the following spectrum ($f_1 = 10 \times f$, $f_2 = 5 \times f$, $f_3 = 1 \times f$).



We can see that the carrier ($n = 0$, $f_0 = 10 \times f$) is not modulated. Then, the next harmonic ($n = 1$, $f_1 = 15 \times f$) is modulated with the index β_3, finally, a third harmonic ($n = 2$, $f_2 = 20 \times f$) is modulated with a double index, $2 \times \beta_3$.

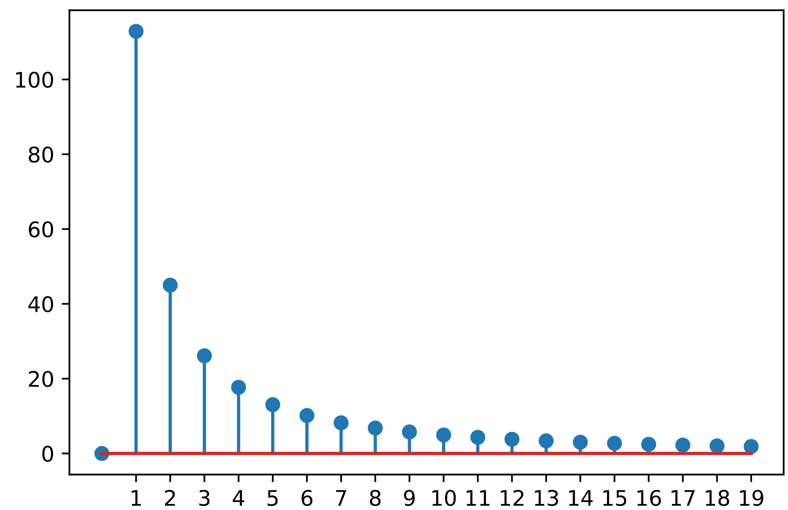
We can expect this configuration to be highly complex, but the DX7 goes one step further with 4 operators stacked in 3 algorithms. The Montage does even propose an algorithm where all of the 8 operators are stacked !

The feedback is also proposed by the DX7 and the Montage : here the output of an operator loops back and modulate itself or an operator above it. Everything happens as if there was an infinite stack with the operators(s) duplicated. It can't be analysed easily[4].

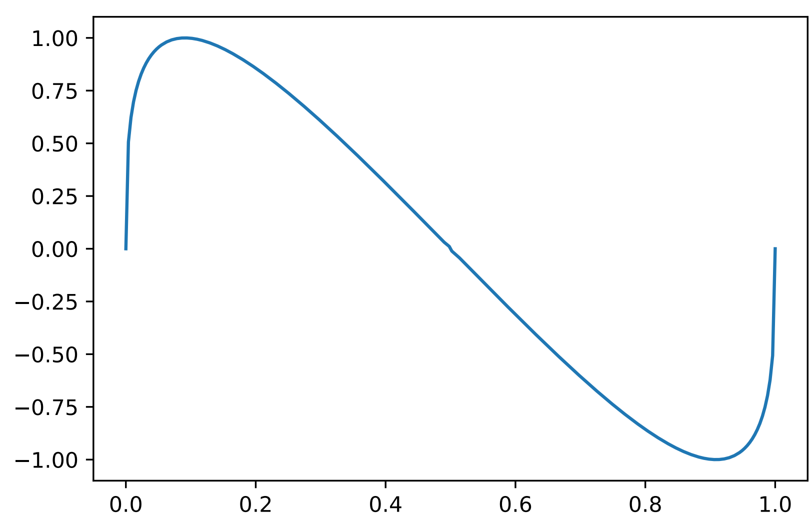
[3] Harmonics of frequencies $f_1 + n \times f_2 + m \times f_3$ have the amplitudes $A_{n,m} = J_n(\beta_2) \times J_m(n\beta_3)$.

[4] However, we can prove that for a single operator, the amplitudes of harmonics at the frequencies $n \times f$ are $A_n = \frac{2J_n(n\beta)}{n\beta}$.

The following graph shows the spectrum of a single operator with a feedback. We can see that the decrease is quite regular.



Its temporal graph is closed to a sawtooth signal proposed by most analog synthesizer :



5 Some examples

We have seen the basic principles of frequency modulation. Let's experiment this principles.

These examples which will be proposed are available with the corresponding sounds at the following URL :

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http://www.sinerj.org/~loyer/FM
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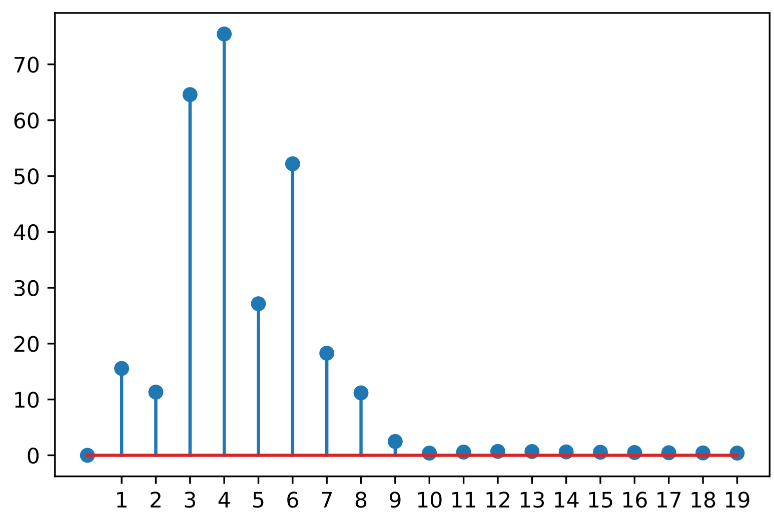
Example 1 If we seek a sound with harmonics up to the rank 6, we can modulate a carrier at frequency $f_c = 1f$ with a modulator at the same frequency, $f_m = 1f$ and a modulation index $\beta = 6$ (output level 90 on a DX7). We will have a sound with a sharp decrease and some hole in the spectrum (because of the oscillations of the Bessel functions and the phase inversion which had been evoked earier).



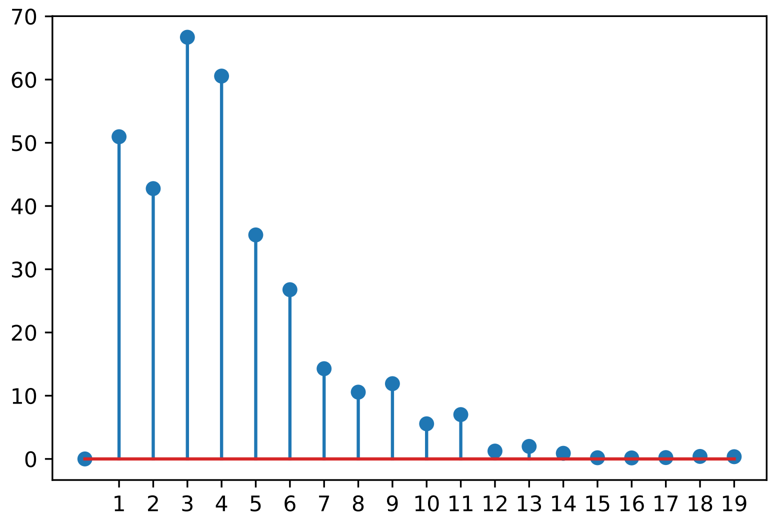
Example 2 If we want a more "natural" sound, we can try to fill the holes and a have a smooth transition. One way is too use a small modulation indexes, near $\beta \approx 1$ but multiple modulators at frequency $4f$, $2f$, $1f$. The modulator at frequency $4f$, will generate high frequency harmonics, but there will be holes. The modulators with lower frequencies will fill them.

Let's try the following values (output levels are from a DX7 table) :

- modulator 1 : $4f$, $\beta = 1$ (level=69)
- modulator 2 : $2f$, $\beta = 0.6$ (level=63)

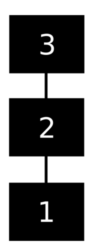
- modulator 3 : $1f$, $\beta = 1.5$ (level=74)

We will find the following spectrum which is more regular :



Example 3 If we want to spare one operator, one way is to stack 3 operators. The two top operators will act as multiple (more than 2) added operators. And we can get something comparable to the previous example.

With the convention :

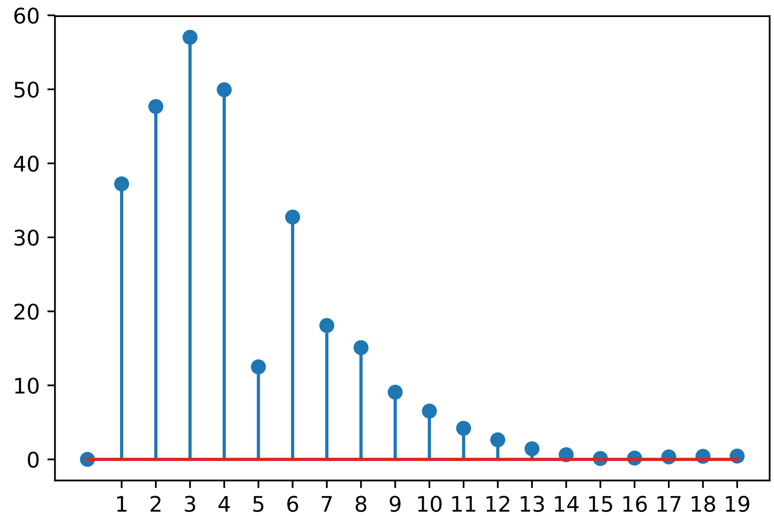


Let's use :

- operator 1 : $f_1 = 2f$
- operator 2 : $f_2 = 1f$, $\beta_2 = 2$ (level 77)

- operator 3 : $f_3 = 1f$, $\beta_3 = 2$ (level 77)

Now, we will have the spectrum :



The spectrum is decreasing quite smoothly, like the previous example. But since we have less modulation indexes to tune, it is more difficult to find values which avoid an hole (like the 5th harmonic in this example). The sounds are however quite closed.

6 Conclusion

We have seen that frequency modulation can generate multiple harmonics with few operators. Each operators has a tunable output level. On modern FM synthesizer, the actual level can be modulated by multiple values : the aftertouch, an envelope, a low frequency oscillator (LFO), which enhance the expressiveness of the synthesizer.

However, the way operator harmonics are combined can be quite complex which makes the FM sound design not obvious.